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LETTERS TO THE EDITOR

ON THE EIGENCHARACTERISTICS OF LONGITUDINALLY VIBRATING RODS CARRYING A TIP MASS AND VISCOUSLY DAMPED SPRING-MASS IN-SPAN

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1. INTRODUCTION

The first author has investigated the eigevalue determination problem of systems obtained by adding spring–mass (damper) secondary systems to laterally vibrating beams and as presented the results in a series of papers. The combined system considered in reference [1] has a spring–mass system attached at its free end which carried tip mass, the other end being fixed. Reference [2] is a more generalized version of reference [1] because there, more than one secondary system is considered. In latter work [3] the case of adding viscous dampers to the secondary systems is investigated. Motivated by the fact that the problem of longitudinally vibrating rods with secondary systems attached has not been invesigated in the literature, the study in reference [4] was considered with the derivation of the eigenfrequencies of a fixed-free longitudinally vibrating elastic rod carrying a tip mass (primary system) to which a spring–mass (secondary system) is attached in-span, and their sensitivity. The present note is in some sense an extention of reference [4] because it is aimed here to derive the characteristic equation for the case in which the attached secondary system is viscously damped. First the "exact" characteristic equation is derived via a boundary value problem formulation. Then, for comparison purposes, a second formulation of the characteristic equation is given for the approximate determination of the characteristic values of the mechanical system.

2. EXACT CHARACTERISTICS EQUATION

The system to be dealt with in the present study is shown in Figure 1. It is a longitudinally vibrating fixed-free elastic rod of axial rigidity EA and mass per unit length m carrying a tip mass M to which a secondary, viscously damped spring–mass system is attached in-span. The main subject of the present study is to derive the exact characteristic equation of the combined system described above in order to determine their eigencharacteristics, i.e., eigenvalues.

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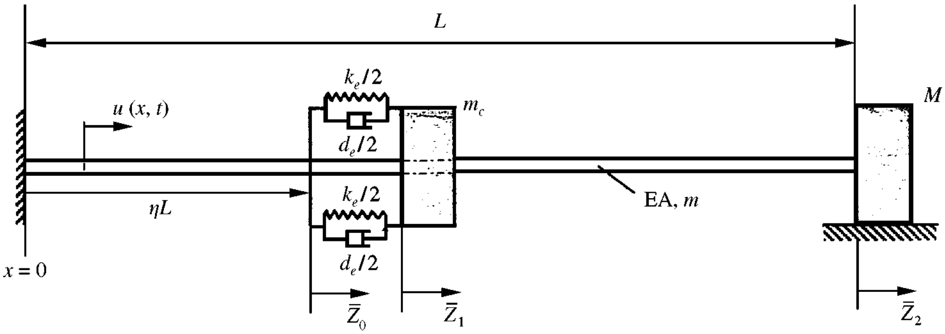

Figure 1. Longitudinally vibrating fixed-free rod with a tip mass and viscously damped spring–mass attached in-span.

The partial differential equation of free longitudinal vibrations of a uniform rod is the well-known equation

$$\mathrm{EA}u''(x, t) = m\ddot{u}(x, t), \tag{1}$$

where $u(x, t)$ represents the axial displacement of the rod at point x and time t. The primes and overdots denote partial derivatives with respect to x and t, respectively. The regions to the left and right of the attachment point of the secondary system to the rod are denoted hereafter as $u_1(x, t)$ and $u_2(x, t)$ where both are subject to differential equation (1). The corresponding boundary and matching conditions are as follows:

$$\begin{gathered} u_1(0, t) = 0, \\ u_1(\eta L, t) = u_2(\eta L, t), \\ \mathrm{EA}u_1'(\eta L, t) - \mathrm{EA}u_2'(\eta L, t) + m_e\,\ddot{\bar{z}}_1 = 0, \\ m_e\ddot{\bar{z}}_1 + k_e\,[\bar{z}_1 - u_1(\eta L, t) + d_e\,[\dot{\bar{z}} - \dot{u}_1(\eta L, t)] = 0, \\ \mathrm{EA}u_2'(L, t) + M\ddot{u}_2(L, t) = 0. \end{gathered} \tag{2}$$

Here, $\bar{z}_1(t)$ means the axial displacements of the mass m_e.

One assumes the solutions to be of the form

$$u_j(x, t) = U_j(x)\,\mathrm{e}^{\lambda t}, \qquad \bar{z}_1(t) = \bar{Z}_1\,\mathrm{e}^{\lambda t}, \quad (j = 1{,}2), \tag{3}$$

where λ denotes the unknown characteristic value of the combined system which is a complex number in general. In the expressions above, both $u_j(x, t)$ and $U_j(x)$ represent complex-valued functions. The essential point here is to imagine the actual longitudinal displacement $u_j(x, t)$ as the real parts of some complex-valued functions for which the same notation is used for the sake of briefness.

By putting expressions (3) into partial differential equation (1), the following ordinary differential equations for functions $U_j(x)$ are obtained:

$$U_j''(x) - \beta^2\,U_j(x) = 0, \quad (j = 1{,}2), \tag{4}$$

where

$$\beta^2 = \frac{m\lambda^2}{\mathrm{EA}} \tag{5}$$

is introduced.

The general solutions of differential equations (4) are

$$U_1(x) = C_1\,\mathrm{e}^{\beta x} + C_2\,\mathrm{e}^{-\beta x},$$
$$U_2(x) = C_3\,\mathrm{e}^{\beta x} + C_4\,\mathrm{e}^{-\beta x}, \tag{6}$$

where $C_1 - C_4$ represent four integration constants yet to be determined. Substitution of the expressions in equation (6) into the boundary and matching conditions (2) yields a set of five equations for the determination of these constants and $\bar{Z}_1$. A non-trivial solution of this set is possible only if the determinant of the coefficients vanishes. This determinant can be brought, after lengthy operations, into the form

$$\begin{aligned} \alpha_{m_e}\bar{\beta}(1+\psi\bar{\beta})(\mathrm{e}^{\eta\bar{\beta}} - \mathrm{e}^{-\eta\bar{\beta}})\{(\beta_M\bar{\beta}-1)(\mathrm{e}^{\eta\bar{\beta}} - \mathrm{e}^{-\eta\bar{\beta}})\,\mathrm{e}^{(\eta-1)\bar{\beta}} - [(1+\beta_M\bar{\beta})\mathrm{e}^{\bar{\beta}} \\ + (1-\beta_M\bar{\beta})\mathrm{e}^{-\bar{\beta}}] - 2\mathrm{e}^{\eta\bar{\beta}}[(1+\beta_M\bar{\beta})\mathrm{e}^{\bar{\beta}} \\ - (\beta_M\bar{\beta}-1)\mathrm{e}^{-\bar{\beta}}]\left(\frac{\alpha_{m_e}}{\alpha_{k_e}}\bar{\beta}^2 + \psi\bar{\beta} + 1\right) = 0, \end{aligned} \tag{7}$$

where the following abbreviations are introduced:

$$\bar{\beta} = \beta L, \qquad \beta_M = \frac{M}{mL}, \qquad \alpha_{m_e} = \frac{m_e}{mL}, \qquad \alpha_{k_e} = \frac{k}{\mathrm{EA}/L},$$
$$D_e = \frac{d}{2m_e\,\omega_e}, \qquad \omega_e^2 = \frac{k_e}{m_e}, \qquad \omega_o^2 = \frac{\mathrm{EA}}{mL^2}, \qquad \psi = \pm 2D_e\frac{\omega_o}{\omega_e} \tag{8}$$

Equation (7) is the characteristic equation of the mechanical system in Figure 1. The solution of this equation with respect to $\bar{\beta}$ yields via

$$\lambda = \mp\,\omega_o\,\bar{\beta}, \tag{9}$$

the "exact" values of the unknown complex eigenvalues λ of the mechanical system.

For comparison of the numerical results, in the following, a second formulation of the characteristic equation of the same system will be given for the approximate determination of the characteristic values. Thus, on the other hand, trial values for the numerical solution of the exact equation can be determined in a very sensitive manner. The formulation is based on the discretization of the rod by its first n eigenfunctions, according to the assumed modes method.

3. AN APPROXIMATE CHARACTERISTIC EQUATION

It was shown in reference [4] that the kinetic and potential energies of the system in Figure 1 can be formulated as

$$T = \frac{1}{2}\sum_{i=1}^{n}\dot{\eta}_i^2 + \frac{1}{2}m_e\,\dot{\bar{z}}_1^2 + \frac{1}{2}M\dot{\bar{z}}_2^2, \qquad V = \frac{1}{2}\sum_{i=1}^{n}\omega_i^2\eta_i^2 + \frac{1}{2}k_e(\bar{z}_1 - \bar{z}_0)^2. \tag{10}$$

Here, $\eta_i(t)$ $(i = 1, \ldots, n)$ are the generalized co-ordinates. $\bar{z}_0(t)$, $\bar{z}_1(t)$ and $\bar{z}_2(t)$ denote the axial displacements of the attachment point of the secondary system to the rod, the secondary mass m_e and the tip mass M, respectively. Finally, ω_i is the ith eigenfrequency of the bare fixed-free elastic rod.

The Rayleigh's dissipation function is

$$F = \frac{1}{2} d_e (\dot{\bar{z}}_1 - \dot{\bar{z}}_0)^2 \tag{11}$$

The formulation of the approximate characteristic eqauation uses the approach of Dowell [5] which was also used in reference [4]. It is essentially based on the assumed modes method in conjunction with Lagrange multipliers method. The result is a determinantial equation for the characteristic equation of the system. Hence, the eigenvalues of the system are determined by solving this equation numerically.

For a system with n degrees of freedom where v redundant co-ordinates are used, Lagrange's equations in connection with Lagrange multipliers are as follows [6]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial F}{\partial \dot{q}_k} + \sum_{l=1}^{v} \lambda_1 \frac{\partial f_1}{\partial q_k}, \quad (k = 1, \ldots, n + v) \tag{12}$$

with the kinetic potential $L = T - V$ and v constraint equations

$$f_1(t; q_1, \ldots, q_{n+v}) = 0, \quad (1 = 1, \ldots, v), \tag{13}$$

Here, q_k and λ_1 denote the kth generalized co-ordinate of the system and lth Lagrangian multiplier. In the present case, there are two constraint equations

$$f_1 := \sum_{k=1}^{n} U_k(L)\eta_k(t) - \bar{z}_2(t) = 0, \tag{14}$$

$$f_2 := \sum_{k=1}^{n} U_k(L)\eta_k(t) - \bar{z}_0(t) = 0,$$

where $U_k(x)$ and $\eta_k(t)$ represent the kth orthonormalized eigenfunction of the bare fixed-free elastic rod and kth generalized co-ordinate, respectively;

$$U_k(x) = \sqrt{\frac{2}{mL}} \sin(2k-1)\frac{\pi x}{2L}, \qquad (k = 1, \ldots, n). \tag{15}$$

The evaluation of Lagrange's equations (12) by considering equations (10), (11), (13) and (14) results in the following set of $n + 3$ equations:

$$\ddot{\eta}_k + \omega_k^2 \eta_k = \lambda_1 U_k(L) + \lambda_2 U_k(\eta L), \ (k = 1, \ldots, n),$$

$$d_e(\dot{\bar{z}}_1 - \dot{\bar{z}}_0) + k_e(\bar{z}_1 - \bar{z}_0) = \lambda_2,$$

$$m_e \ddot{\bar{z}}_1 + d_e(\dot{\bar{z}} - \dot{\bar{z}}_o) + k_e(\bar{z}_1 - \bar{z}_0) = 0,$$

$$M\ddot{\bar{z}}_2 = -\lambda_1. \tag{16}$$

The substitution of the exponential solutions

$$\eta_k = \bar{\bar{\eta}}_k\, e^{\lambda t},\ (k = 1, \ldots, n), \qquad \bar{z}_0 = \bar{\bar{z}}_0\, e^{\lambda t}, \qquad \bar{z}_1 = \bar{\bar{z}}_1\, e^{\lambda t},$$
$$\bar{z}_2 = \bar{\bar{z}}_2\, e^{\lambda t}, \qquad \lambda_1 = \bar{\bar{\lambda}}_1\, e^{\lambda t}, \qquad \lambda_2 = \bar{\bar{\lambda}}_2\, e^{\lambda t} \tag{17}$$

into equations (14) and (16) yields a set of $(n + 5)$ equations for the amplitudes of the exponential functions. It can be shown that these equations result in a set of two homogeneous equations for $\bar{\bar{\lambda}}_1$ and $\bar{\bar{\lambda}}_2$. A non-trivial solution of this set is possible only if the determinant of the coefficients vanishes. This in turn leads to the following characteristic equation of the system in Figure 1:

$$\left[\sum_{k=1}^{n} \frac{U_k^2(L)}{\lambda^2 + \omega_k^2} + \frac{1}{M\lambda^2}\right]\left[\sum_{k=1}^{n} \frac{U_k^2(\eta L)}{\lambda^2 + \omega_k^2} + \frac{m_e\lambda^2 + d_e\lambda + k_e}{m_e\lambda^2\,(d_e\lambda + k_e)}\right]$$
$$-\left[\sum_{k=1}^{n} \frac{U_k(L)\,U_k(\eta L)}{\lambda^2 + \omega_k^2}\right]^2 = 0. \tag{18}$$

Using the abbreviations

$$a_k = \sqrt{2}\,(-1)^{k+1}, \qquad b_k(\eta) = \sqrt{2}\,\sin\,(2k - 1)\,\frac{\pi}{2}\,\eta,$$
$$\lambda_k = \left[(2k - 1)\,\frac{\pi}{2}\right]^2, \qquad \omega_o^2 = \frac{EA}{mL^2}, \qquad \lambda^* = \frac{\lambda}{\omega_o} \tag{19}$$

and those defined in equation (8), the characteristic equation above can be rewritten in terms of non-dimensional quantities as

$$\left[\sum_{k=1}^{n} \frac{a_k^2}{\lambda^{*2} + \lambda_k} + \frac{1}{\beta_M\,\lambda^{*2}}\right]\left[\sum_{k=1}^{n} \frac{b_k^2}{\lambda^{*2} + \lambda_k} + \frac{\lambda^{*2} + 2D_e\sqrt{\alpha_{k_e}/\alpha_{m_e}}\,\lambda^* + \alpha_{k_e}/\alpha_{m_e}}{\lambda^{*2}\,(2D_e\sqrt{\alpha_{k_e}\,\alpha_{m_e}}\,\lambda^* + \alpha_{k_e})}\right]$$
$$-\left[\sum_{k=1}^{n} \frac{a_k\,b_k}{\lambda^{*2} + \lambda_k}\right]^2 = 0. \tag{20}$$

This equation is to be solved with respect of λ^*. It is reasonable to expect that the dimensionless characteristics values λ^* obtained from equation (20) converge to those of the exact characteristic equation (7) if n goes to infinity. It is to be noted that according to equation (9), $\lambda^* = \pm\bar{\beta}$.

4. NUMERICAL RESULTS

This section is devoted to the numerical evaluation of the expressions established in the preceding sections. For the numerical applications, the following non-dimensional values are chosen for the physical data of the mechanical system in Figure 1: $\alpha_{m_e} = \alpha_{k_e} = 1$, $D_e = 0{\cdot}025$. The number of the modes n in expansion (10) is chosen is 100. The first five pairs of dimensionless eigenvalues λ^* of the system (arranged with respect to the magnitude of the imaginary parts) are given in Table 1 as a function of η, i.e. the location of the attachment point of the secondary

TABLE 1

Dimensionless eigenvalues λ^* of the system in Figure 1 for various values of η, i.e. location parameter of the attachment point of the secondary system to the rod $\alpha_{m_e} = \alpha_{k_e} = 1$, $D_e = 0{\cdot}025$, $\beta_M = 0{\cdot}5$

η = 0·1	0·4	0·6	0·8	1·0
− 0·016522 ± 0·934686i	− 0·007054 ± 0·779786i	− 0·004916 ± 0·714694i	− 0·003745 ± 0·667270i	− 0·003042 ± 0·631719i
− 0·016543 ± 0·935074i	− 0·007060 ± 0·779997i	− 0·004920 ± 0·714856i	− 0·003747 ± 0·667474i	− 0·003048 ± 0·632049i
− 0·004928 ± 1·104681i	− 0·014501 ± 1·263212i	− 0·022880 ± 1·398754i	− 0·034797 ± 1·547635i	− 0·045130 ± 1·669049i
− 0·004948 ± 1·105107i	− 0·014546 ± 1·263929i	− 0·022960 ± 1·399752i	− 0·034933 ± 1·548949i	− 0·045142 ± 1·669058i
− 0·005628 ± 3·674555i	− 0·046034 ± 3·884476i	− 0·031699 ± 3·809484i	− 0·002487 ± 3·656587i	− 0·014384 ± 3·709847i
− 0·005665 ± 3·679770i	− 0·046260 ± 3·890118i	− 0·031676 ± 3·814111i	− 0·002412 ± 3·661240i	− 0·014284 ± 3·714404i
− 0·016523 ± 6·630694i	− 0·012251 ± 6·614116i	− 0·026273 ± 6·655554i	− 0·033208 ± 6·681310i	− 0·004598 ± 6·591361i
− 0·016643 ± 6·642612i	− 0·012093 ± 6·625251i	− 0·026741 ± 6·668383i	− 0·032935 ± 6·691918i	− 0·004556 ± 6·602913i
− 0·030902 ± 9·696167i	− 0·020675 ± 9·672601i	− 0·011510 ± 9·653408i	− 0·049043 ± 9·729433i	− 0.002154 ± 9·633866i
− 0·031114 ± 9·714708i	− 0·021115 ± 9·691676i	− 0·011104 ± 9·670791i	− 0·049372 ± 9·748188i	− 0·002133 ± 9·652114i

TABLE 2

Dimensionless eigevalues λ^* of the system in Figure 1 for various values of the tip mass parameter β_M. $\alpha_{m_e} = \alpha_{k_e} = 1$, $D_e = 0{\cdot}025$, $\eta = 0{\cdot}4$

β_M			
0·25	0·50	0·75	1·0
– 0·008709 ± 0·799660i	– 0·007054 ± 0·779786i	– 0·005276 ± 0·755199i	– 0·036647 ± 0·726830i
– 0·008717 ± 0·799886i	– 0·007060 ± 0·779997i	– 0·005282 ± 0·755411i	– 0·003671 ± 0·727059i
– 0·016098 ± 1·461581i	– 0·014501 ± 1·263212i	– 0·014664 ± 1·145315i	– 0·015334 ± 1·072404i
– 0·016143 ± 1·462082i	– 0·014546 ± 1·263929i	– 0·014707 ± 1·146120i	– 0·015375 ± 1·073228i
– 0·046515 ± 4·160377i	– 0·046034 ± 3·884476i	– 0·045982 + 3·755257i	– 0·046002 ± 3·681712i
– 0·046702 ± 4·164294i	– 0·046260 ± 3·890118i	– 0·046223 ± 3·761531i	– 0·046250 ± 3·688277i
– 0·008190 ± 6·837156i	– 0·012251 ± 6·614116i	– 0·014067 ± 6·527590i	– 0·015072 ± 6·482264i
– 0·008079 ± 6·846405i	– 0·012093 ± 6·625251i	– 0·013894 ± 6·539198i	– 0·014893 ± 6·494050i
– 0·023804 ± 9·860570i	– 0·020675 ± 9·672601i	– 0·019549 ± 9·604263i	– 0·018973 ± 9·569215i
– 0·024223 ± 9·877666i	– 0·021115 ± 9·691676i	– 0·019999 ± 9·623787i	– 0·019414 ± 9·588908i

system to the rod, where $\beta_M = 0{\cdot}5$ is chosen. The complex numbers in the first rows are λ^*-values obtained from the numerical solution of the "exact" characteristic equation (7). The numbers in the second rows are values obtained from the approximate equation (20) considering equations (9) and (19). The numerical solutions of Equations (7) and (20) are carried out by MATHEMATICA. The agreement of "exact" and approximate characteristic values is very good.

As another numerical application, in Table 2, the dimensionless characteristic values λ^* are given as a function of β_M, i.e., dimensionless tip mass ratio, where $\eta = 0{\cdot}4$ is chosen. The comparison of the complex numbers from both rows indicates clearly that the agreement is again very good. During the numerical computations, it is observed that only in the case of the (+) sign of ψ, as given by equation (8), physically meaningful characteristic values are obtained.

5. CONCLUSIONS

The present study deals with the determination of the characteristic values of a fixed-free, longitudinally vibrating elastic rod carrying a tip mass to which a viscously damped spring–mass system is attached in-span. The "exact" characteristic equation is established via a boundary value problem formulation. Moreover, for comparison purposes, using the normal mode method, an approximate but quite accurate characteristic equation is established. Both characteristic equations are then numerically solved for various combinations of physical parameters. The results are collected in two tables.

REFERENCES

1. M. Gürgöze 1998 *Journal of Sound and Vibration* **190**, 149–162. On the eigenfrequencies of a cantilever beam with attached tip mass and a spring–mass system.
2. M. Gürgöze 1998 *Journal of Sound and Vibration* **217**, 585–595. On the alternative formulations of the frequency equation of a Bernoulli–Euler beam to which several spring–mass systems are attached in-span.
3. M. Gürgöze 1999 *Journal of Sound and Vibration.* **223**, 666–677 Alternative formulations of the characteristic equation of a Bernoulli–Euler beam to which several viscously damped spring–mass systems are attached in-span.
4. M. Gürgöze 1998 *Journal of Sound and Vibration*, **216**, 295–308. On the eigenfrequencies of longitudinally vibrating rods carrying a tip mass and spring–mass in-span.
5. E.H. Dowell 1979 *Journal of Applied Mechanics* **46**, 206–209. On some general properties of combined dynamical systems.
6. L. Meirovitch 1967 *Methods of Analytical Dynamics.* New York: McGraw-Hill